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SPACES OF VECTOR MEASURES

BY

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ABSTRACT. Let $C_{rc} = C_{rc}(X, E)$ denote the space of all continuous functions f, from a completely regular Hausdorff space X into a locally convex space E, for which $f(X)$ is relatively compact. As it is shown in [8], the uniform dual C'_{rc} of C_{rc} can be identified with a space $M(B, E')$ of E'-valued measures defined on the algebra of subsets of X generated by the zero sets. In this paper the subspaces of all σ-additive and all τ-additive members of $M(B, E')$ are studied. Two locally convex topologies β and β_1 are considered on C_{rc}. They yield as dual spaces the spaces of all τ-additive and all σ-additive members of $M(B, E')$ respectively. In case E is a locally convex lattice, the σ-additive and τ-additive members of $M(B, E')$ correspond to the σ-additive and τ-additive members of C_{rc} respectively.

1. **Definitions and preliminaries.** Let X be a completely Hausdorff space and let E be a real locally convex Hausdorff space. Let $C^b = C^b(X)$ denote the space of all bounded continuous real-valued functions on X. We will denote by $C_{rc} = C_{rc}(X, E)$ the space of all continuous functions f, from X into E, for which $f(X)$ is relatively compact. Clearly C_{rc} consists of those continuous functions f, from X into E, that have continuous extensions $\hat{f}$ to all of the Stone-Čech compactification βX of X. For an f in C^b we will denote also by $\hat{f}$ its unique continuous extension to all of βX. The zero sets in X are defined to be the kernels of real continuous functions on X. The complement of a zero set is called a cozero set.

Let Σ be an algebra of subsets of X and let m be a finitely-additive bounded real set function on Σ. We say that m is regular with respect to a subfamily Σ_1 of Σ if the following condition is satisfied: For every F in Σ and every $\epsilon > 0$ there exists G in Σ_1 such that $G \subset F$ and $|m(H)| < \epsilon$ for all H in Σ which are contained in $F - G$.

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Let now $B = B(X)$ and $Ba = Ba(X)$ be, respectively, the algebra and σ-algebra of subsets of X generated by the zero sets. The collection of Borel subsets of X will be denoted by $Bo = Bo(X)$. Let $M(X)$ be the space of all bounded, finitely-additive, real-valued, set functions on $B(X)$ which are regular with respect to the family of zero sets. The space of all bounded, countably-additive, real-valued, regular (with respect to the zero sets) measures on Ba will be denoted by $M_\sigma(Ba)$. By $M_\tau(Bo)$ we will denote the space of all real, regular with respect to the closed sets, Borel measures m on Bo such that $|m|(Z_\alpha) \longrightarrow 0$ for each net $\{Z_\alpha\}$ of zero sets which decreases to the empty set (see Varadarajan [17] or Aleksandrov [1]). Note that an element m of $M_\tau(Bo)$ is not necessarily regular with respect to the zero sets. However its restriction to Ba is an element of $M_\sigma(Ba)$ by Varadarajan [16, p. 171, Theorem 19]. The subspaces of all σ-additive and τ-additive members of $M(X)$ will be denoted by $M_\sigma(X)$ and $M_\tau(X)$ respectively (see Varadarajan [17] for the definition of σ-additive and τ-additive measures). For m in any one of the spaces $M(X)$, $M_\sigma(Ba)$, $M_\tau(Bo)$, the positive part m^+, the negative part m^-, and the variation $|m|$ are understood as, for example, in Aleksandrov [1].

Let now $\{p: p \in I\}$ be a family of continuous seminorms on E generating the topology of E. We choose this family so that it is directed, i.e., given p_1, p_2 in I there exists $p \in I$ with $p \geqslant p_1, p_2$. For each p in I we consider the space $M_p(B, E')$ of all finitely-additive functions $m: B(X) \longrightarrow E'$ (E' is the topological dual of E) such that the following two conditions are satisfied:

(1) For each $s \in E$, the function $ms: B \longrightarrow R$, $(ms)(F) = m(F)s$, is in $M(X)$.

(2) $\|m\|_p = m_p(X) < \infty$, where for F in B we define $m_p(F) = \sup|\Sigma\, m(F_i)s_i|$ the supremum being taken over all finite B-partitions $\{F_i\}$ of F (that is partitions into sets in B) and all finite collections $\{s_i\}$ in E with $p(s_i) \leqslant 1$.

The set function m_p belongs to $M(X)$. Indeed it is easy to see that m is finitely-additive and bounded. For the regularity, consider an F in B and let $\epsilon > 0$ be given. By definition there exist a finite B-partition $\{F_i\}$ of F and $s_i \in E$, with $p(s_i) \leqslant 1$, such that $\Sigma m(F_i)s_i > m_p(F) - \epsilon$. By the regularity of ms_i we can choose for each i a zero set $Z_i \subset F_i$ such that $\Sigma m(Z_i)s_i > m_p(F) - \epsilon$. The zero set $Z = \bigcup Z_i$ is contained in F. Moreover we have $m_p(Z) \geqslant \Sigma m(Z_i)s_i > m_p(F) - \epsilon$. This proves the regularity of m_p. Set $M(B, E') = \bigcup_{p \in I} M_p(B, E')$.

Let σ denote the uniform topology on C_{rc}, i.e., the locally convex topology generated by the family of seminorms $\{\|\cdot\|_p: p \in I\}$, where $\|f\|_p = \sup\{p(f(x)): x \in X\}$. In [8] the author defines the integral of a function f in C_{rc} with respect to a member of $M(B, E')$. The integration process employed is a generalization of the process of Aleksandrov to the vector case. It is one of the many integration processes defined by McShane [12]. Every element m of

$M(B, E')$ generates a linear functional ϕ_m on C_{rc} by $\phi_m(f) = \int_X f dm$, $f \in C_{rc}$. The proof of the following theorem can be found in [8].

THEOREM 1.1. *For each* $m \in M(B, E')$, ϕ_m *is an element of* $(C_{rc}, \sigma)' = C'_{rc}$. *Moreover, the map* $m \longrightarrow \phi_m$, *from* $M(B, E')$ *into* C'_{rc}, *is linear, one-to-one, and onto.*

THEOREM 1.2. *If* $m \in M_p(B, E')$, *then* $\|\phi_m\| = \|m\|_p$, *where* $\|\phi_m\|_p = \sup\{|\phi_m(f)|: f \in C_{rc}, \|f\|_p \leqslant 1\}$.

PROOF. It is clear from the definitions that $|\int f dm| \leqslant \int p \circ f\, dm_p \leqslant \|f\|_p \|m\|_p$ for all C_{rc} and hence $\|\phi_m\|_p \leqslant \|m\|_p$. On the other hand, let $\epsilon > 0$ be given. By the definition of $\|m\|_p$, there exist a finite B-partition $\{F_i\}$ of X and $s_i \in E$ with $p(s_i) \leqslant 1$ such that $\|m\|_p < \Sigma m(F_i)s_i + \epsilon$. By regularity there are zero sets $Z_i \subset F_i$ such that $\|m\|_p < \Sigma m(Z_i)s_i + \epsilon$. Again by the regularity of ms_i we can find pairwise disjoint cozero sets $\{U_i\}$, $Z_i \subset U_i$, such that

$$\sum |ms_i|(U_i - Z_i) < \epsilon.$$

For each i choose h_i in C^b, $0 \leqslant h_i \leqslant 1$, such that $h_i = 1$ on Z_i and $h_i = 0$ on $X - U_i$. Set $h = \Sigma h_i s_i$. Then $\|h\|_p \leqslant 1$ and so $|\int h\, dm| \leqslant \|\phi\|_p$. But

$$\left|\int h dm\right| \geqslant \left|\sum \int_{Z_i} s_i dm\right| - \left|\sum \int_{U_i - Z_i} h_i d(ms_i)\right| \geqslant \sum m(Z_i)s_i - \epsilon > \|m\|_p - 2\epsilon.$$

Since $\epsilon > 0$ was arbitrary we get that $\|\phi_m\|_p \geqslant \|m\|_p$ and this completes the proof.

In case E is a locally convex lattice, (C_{rc}, σ) becomes also a locally convex lattice under the pointwise ordering (that is, we define $f \geqslant g$ iff $f(x) \geqslant g(x)$ for all $x \in X$). We define an order relation $\geqslant$ on $M(B, E')$ by $m_1 \geqslant m_2$ iff $m_1(F) \geqslant m_2(F)$ for all F in B. Note that E' is a lattice when ordered by the cone $\{\phi \in E': \phi(s) \geqslant 0$ when $s \geqslant 0\}$. As it is shown in [8], $M(B, E')$ becomes a lattice and the map $m \longrightarrow \phi_m$, of Theorem 1.1, is lattice preserving.

2. **Extensions of members of** $M(B, E')$. Let $p \in I$. We define $M_{\sigma,p}(Ba, E')$ to be the set of all functions m: $Ba \longrightarrow E'$ such that the following two conditions are satisfied:

(1) For each s in E the function ms: $Ba \longrightarrow R$, $(ms)(F) = m(F)s$, is in $M_s(Ba)$.

(2) $m_p(X) < \infty$ where, for each F in Ba, we define $m_p(F) = \sup|\Sigma m(F_i)s_i|$ where the supremum is taken over all finite Ba-partitions $\{F_i\}$ of F and all finite collections $\{s_i\}$ in E with $p(s_i) \leqslant 1$.

LEMMA 2.1. *If* $m \in M_{\sigma,p}(Ba, E')$, *then* $m_p \in M_\sigma(Ba)$.

PROOF. It is easy to see that m_p is bounded monotone and finitely-additive. Let $\{F_n\}$ be a sequence of pairwise Baire sets (i.e., sets in Ba) and set $F = \bigcup F_n$. Since m_p is monotone and finitely-additive, we have $m_p(F) \geq m_p(\bigcup_1^n F_i) = \Sigma_1^n m_p(F_i)$ for each n. Hence $m_p(F) \geq \Sigma m_p(F_i)$. On the other hand, let $\epsilon > 0$ be arbitrary. There exist a Ba-partition $G_1, \cdots, G_N$ of F and $s_i \in E$, $p(s_i) \leq 1$, such that $\Sigma_1^N m(G_i)s_i > m_p(F) - \epsilon$. Since ms_i is countably additive we have $m(G_i)s_i = \Sigma_{n=1}^{\infty} m(G_i \cap F_n)s_i$. Moreover,

$$\sum_{n=1}^{\infty} \sum_{i=1}^{N} |m(G_i \cap F_n)s_i| \leq \sum_{n=1}^{\infty} m_p(F_n) \leq m_p(F) < \infty.$$

Hence

$$m_p(F) - \epsilon \leq \sum_{i=1}^{N} m(G_i)s_i = \sum_{i=1}^{N} \sum_{n=1}^{\infty} m(G_i \cap F_n)s_i$$

$$= \sum_{n=1}^{\infty} \sum_{i=1}^{N} m(G_i \cap F_n)s_i \leq \sum m_p(F_n) \leq m_p(F).$$

Since $\epsilon > 0$ was arbitrary we conclude that $m_p(F) = \Sigma m_p(F_n)$ and so m_p is countably-additive. Finally, the proof of the regularity of m_p is similar to that of the case of a member of $M_p(B, E')$.

Next we define $M_{\tau,p}(B, E')$ to be the set of all $m: Bo \longrightarrow E'$ having the following two properties:

(a) For each s in E, ms belongs to $M_\tau(Bo)$.

(b) $m_p(X) < \infty$, where for each F in Bo the $m_p(F)$ is defined by $m_p(F) = \sup|\Sigma m(F_i)s_i|$ the supremum being taken over all finite Bo-partitions of F and all finite collections $\{s_i\}$ in E with $p(s_i) \leq 1$.

LEMMA 2.2. *If $m \in M_{\tau,p}(Bo, E')$, then $m_p \in M_\tau(Bo)$.*

PROOF. By using an argument similar to that of 2.1, we show that m_p is a bounded, countably-additive, regular with respect to the closed sets, Borel measure on X. To complete the proof we need to show that m_p is τ-additive. To this end, consider an arbitrary net $\{Z_\alpha\}$ of zero sets decreasing to the empty set. For each α there exists a zero set $\hat{Z}_\alpha$ in βX such that $Z_\alpha = \hat{Z}_\alpha \cap X$.

Define $\bar{m}: Bo(\beta X) \longrightarrow E'$ by $\bar{m}(F) = m(F \cap X)$. For each $s \in E$, the function $\bar{m}s: Bo(\beta X) \longrightarrow R$, $(\bar{m}s)(F) = (ms)(F \cap X)$, is a regular Borel measure on βX since ms is τ-additive (see Knowles [11]). It follows now easily that $m \in M_{\tau,p}(Bo(\beta X), E')$. Moreover $\bar{m}_p(F) = m_p(F \cap X)$ for each Borel set F in βX. Indeed it is clear that $\bar{m}_p(F) \leq m_p(F \cap X)$. On the other hand, if $\{G_i\}$ is a finite $Bo(X)$ partition of $F \cap X$, then there are pairwise disjoint Borel sets V_i

in βX, which we may choose contained in F, such that $G_1 = V_i \cap X$. For $s_i \in E$ with $p(s_i) \leqslant 1$, we have $\bar{m}_p(F) \geqslant |\Sigma \bar{m}(V_i)s_i| = |\Sigma m(G_i)s_i|$. This shows that $\bar{m}_p(F) \geqslant m_p(F \cap X)$ and so $\bar{m}_p(F) = m_p(F \cap X)$. Let now $D = \{Z \subset \beta X$: Z is an intersection of a finite number of $\hat{Z}_\alpha$'s$\}$. Then D is directed downwards to $G = \bigcap \hat{Z}_\alpha$. Hence $\bar{m}_p(G) = \lim_{Z \in D} \bar{m}_p(Z)$. Since $G \cap X = \varnothing$, we have $\bar{m}_p(G) = 0$. Thus given $\epsilon > 0$ there exists $Z = \hat{Z}_{\alpha_1} \cap \cdots \cap \hat{Z}_{\alpha_n}$ in D such that $\bar{m}_p(Z) < \epsilon$. If $\alpha \geqslant \alpha_1, \cdots, \alpha_n$, we have $m_p(Z_\alpha) \leqslant m_p(Z \cap X) = \bar{m}_p(Z) < \epsilon$. This completes the proof.

THEOREM 2.3. *If* $m \in M_{\sigma,p}(Ba, E')$ $[m \in M_{\tau,p}(Bo, E')]$, *then* $m_p(X) = \sup\{|\int f\,dm|: f \in C_{rc}, \|f\|_p \leqslant 1\}$.

PROOF. Let $d = \sup\{|\int f\,dm|: f \in C_{rc}, \|f\|_p \leqslant 1\}$. To prove the result in the case of an m in $M_{\sigma,p}(Ba, E')$ one can use the same argument as the one used in the proof of Theorem 1.2. We will prove the result for an m in $M_{\tau,p}(Bo, E')$. Since $|\int f\,dm| < \|f\|_p m_p(X)$, it follows that $d \leqslant m_p(X)$. To prove the reverse inequality, consider an arbitrary $\epsilon > 0$. Define $\bar{m}$ on $Bo(\beta X)$ by $\bar{m}(F) = m(F \cap X)$. As we have seen in the proof of Lemma 2.2, we have $\bar{m} \in M_{\tau,p}(Bo(\beta X), E')$ and $\bar{m}_p(F) = m_p(F \cap X)$ for each Borel set F in βX. By the definition of $\bar{m}_p$, there exist a partition $\{F_1, \cdots, F_n\}$ of βX, $F_i \in Bo(\beta X)$, and $s_i, \cdots, s_n$ in E, $p(s_i) \leqslant 1$, such that $\Sigma \bar{m}(F_i)s_i > m_p(\beta X) - \epsilon = m_p(X) - \epsilon$. By regularity there are closed sets G_i in βX, $G_i \subset F_i$, such that $\Sigma \bar{m}(G_i)s_i > m_p(X) - \epsilon$. Next we choose pairwise disjoint open sets O_i in βX, $G_i \subset O_i$, such that $|\bar{m}s_i|(O_i - G_i) < \epsilon/n$. For each i, $1 \leqslant i \leqslant n$, there is an h_i in $C^b(X)$, $0 \leqslant h_i \leqslant 1$, $\hat{h}_i = 1$ on G_i and $\hat{h}_i = 0$ on the complement of O_i. Set $h = \Sigma h_i s_i$. Then $\|h\|_p \leqslant 1$ and

$$\int_X h\,dm = \int_{\beta X} \hat{h}\,d\bar{m} = \sum \bar{m}(G_i)s_i + \sum \int_{O_i - G_i} \hat{h}_i\,d(\bar{m}s_i) > m_p(X) - 2\epsilon.$$

Thus $d > m_p(X) - 2\epsilon$ and the result follows since $\epsilon > 0$ was arbitrary.

LEMMA 2.4. *Let* $m \in M_{\tau,p}(Bo, E')$ *and* $\mu = m|_{Ba}$ (= *restriction of* m *to* Ba). *Then* (a) $\mu \in M_{\sigma,p}(Ba, E')$, (b) $\mu_p = m_p|_{Ba}$.

PROOF.. Part (a) is clear because the restriction to Ba of an element of $M_\tau(Bo)$ is in $M_\sigma(Ba)$. For (b) we first observe that $\mu_p(X) = m_p(X)$ by 2.3 since $\int f\,dm = \int f\,d\mu$ for all f in C_{rc}. It is also clear that $\mu_p(F) \leqslant m_p(F)$ for all F in Ba. Thus (b) follows.

Set $M_\sigma(Ba, E') = \bigcup\{M_{\sigma,p}(Ba, E'): p \in I\}$ and define $M_\tau(Bo, E')$ analogously.

Let $M_\sigma(B, E')$ be the subspace of $M(B, E')$ consisting of all $m \in M(B, E')$ for which $ms \in M_\sigma(X)$ for all s in E. We define $M_\tau(B, E')$ similarly. We will call the elements of $M_\sigma(B, E')$ $[M_\tau(B, E')]$ the σ-additive (τ-additive) members of $M(B, E')$. The next theorem shows that the σ-additive members of $M(B, E')$ are exactly the ones that have extensions to members of $M_\sigma(Ba, E')$.

THEOREM 2.5. *Let $m \in M(B, E')$. Then m is σ-additive iff there exists a μ in $M_\sigma(Ba, E')$ with $m = \mu|_B$. Moreover, if such a μ exists it is unique.*

PROOF. Clearly $\mu|_B$ is in $M_\sigma(B, E')$ for each μ in $M_\sigma(Ba, E')$. Moreover if λ is another member of $M_\sigma(Ba, E')$ such that $\lambda|_B = \mu|_B$, then $\lambda s|_B = \mu s|_B$ for each $s \in E$. It follows that $\lambda s = \mu s$ by the regularity of λs and μs. This, being true for all s in E, implies that $\mu = \lambda$. Assume next that $m \in M_\sigma(B, E')$. For each $s \in E$, $ms \in M_\sigma(X)$. Hence, for each $s \in E$, there exists a unique extension μ_s of ms to a member of $M_\sigma(Ba)$ such that $\|ms\| = \|\mu_s\|$ (see Varadarajan [17]). For an F in Ba, we define $\mu(F)$: $E \longrightarrow R$, by $\mu(F)s = \mu_s(F)$. Clearly $\mu(F)$ is linear. Moreover, if $m \in M_p(B, E')$, then

$$|\mu(F)s| = |\mu_s(F)| \leqslant \|\mu_s\| = \|ms\| \leqslant p(s)\|m\|_p.$$

Hence $\mu(F) \in E'$. In this way we define a map μ: $Ba \longrightarrow E'$ such that $\mu s = \mu_s \in B_\sigma(Ba)$ for all $s \in E$. To finish the proof it remains to show that $\|\mu\|_p < \infty$. To this end, consider an arbitrary Ba-partition $F_1, \cdots, F_n$ of X and let $s_i \in E$ with $p(s_i) \leqslant 1$. For $\epsilon > 0$, there exist zero sets $Z_i, \cdots, Z_n$, $Z_i \subset F_i$, such that $|\mu s_i|(F_i - Z_i) < \epsilon/n$. Thus

$$\left|\sum \mu(F_i)s_i\right| < \left|\sum \mu(Z_i)s_i\right| + \epsilon = \left|\sum m(Z_i)s_i\right| + \epsilon \leqslant \|m\|_p + \epsilon.$$

It follows that $\|\mu\|_p \leqslant \|m\|_p$ and the proof is complete.

We have an analogous theorem for $M_\tau(Bo, E')$.

THEOREM 2.6. *Let $m \in M(B, E')$. Then m is τ-additive iff there exists a unique $\mu \in M_\tau(Bo, E')$ such that $m = \mu|_B$.*

PROOF. Clearly $\mu|_B \in M_\tau(B, E')$ for each $\mu \in M_\tau(Bo, E')$. Also, if μ_1, μ_2 are both in $M_\tau(Bo, E')$ with $\mu_1|_B = \mu_2|_B$, then $\mu_1 s|_B = \mu_2 s|_B$ for each s in E. By Kirk [9, Theorem 1.14], we have $\mu_1 s = \mu_2 s$. This, being true for all s in E, implies that $\mu_1 = \mu_2$. Assume now that $m \in M_\tau(B, E')$. Let $C(\beta X, E)$ denote the space of all continuous functions from βX into E.

Clearly $C(\beta X, E) = \{\hat{f}: f \in C_{rc}\}$. Define ϕ on $C(\beta X, E)$ by $\phi(\hat{f}) = \int f\, dm$. Then ϕ is continuous with respect to the uniform topology on $C(\beta X, E)$. Hence, by 1.1, there exists $\bar{m} \in M_p(B(\beta X), E')$ such that $\phi(\hat{f}) = \int \hat{f} \bar{m}$ for all f in C_{rc}. Since each $\bar{m}s$, $s \in E$, is τ-additive it has a unique norm-preserving extension to a member $\bar{\mu}_s$ of $M_\tau(Bo(\beta X))$ (see Kirk [9]). For each Borel set F in βX, we define

$\bar{\mu}(F)$ on E by $\bar{\mu}(F)s = \bar{\mu}_s(F)$. It is easy to see that $\bar{\mu}(F) \in E'$. In this way we get a map $\bar{\mu}$: $Bo(\beta X) \longrightarrow E'$. We will show that $\bar{\mu} \in M_{\tau,p}(Bo(\beta X), E')$. Since $\bar{\mu}s = \bar{\mu}_s \in M_\tau(Bo(\beta X))$, it only remains to show that $\|\bar{\mu}\|_p < \infty$. To this end consider an arbitrary partition $F_1, \cdots, F_n$ of βX into Borel sets and let $s_i \in E$ with $p(s_i) \leqslant 1$. There are closed sets $G_1, \cdots, G_n$ in βX, $G_i \subset F_i$, such that $|\bar{\mu}s_i|(F_i - G_i) < \epsilon/n$ ($\epsilon > 0$ arbitrary). Since $G_i, \cdots, G_n$ are pairwise disjoint compact sets and since the cozero sets form a base for the open sets, there are pairwise disjoint cozero sets $U_1, \cdots, U_n$ in βX, $G_i \subset U_i$, such that $|\bar{\mu}s_i|(U_i - G_i) < \epsilon/n$. Thus

$$\left|\sum \bar{\mu}(F_i)s_i\right| \leqslant \left|\sum \bar{\mu}(U_i)s_i\right| + 2\epsilon = \left|\sum \bar{m}(U_i)s_i\right| + 2\epsilon \leqslant \bar{m}_p(\beta X) + 2\epsilon.$$

It follows that $\bar{\mu}_p(\beta X) \leqslant \bar{m}_p(\beta X)$ and so $\bar{\mu}$ is in $M_{\tau,p}(Bo(\beta X), E')$. Next we show that $\bar{\mu}_p(F) = 0$ for each Borel set F in βX which is disjoint from X. By regularity it suffices to show that $\bar{\mu}(F)s = 0$ for each $s \in E$ and each closed set F in βX disjoint from X. So, let F be such a set and let $s \in E$. There exists an open set 0 in βX, $F \subset 0$, such that $|\bar{\mu}s|(O - F) < \epsilon$ ($\epsilon > 0$ arbitrary). There exists a net $\{f_\alpha\}$ in $C^b(X)$, $f_\alpha \downarrow 0$, $\hat{f}_\alpha = 1$ on F and $\hat{f}_\alpha = 0$ on the complement of 0, $0 \leqslant f_\alpha \leqslant 1$. Since ms is τ-additive, we have $\lim \int f_\alpha d(ms) = 0$. Hence there exists α such that $|\int f_\alpha d(ms)| < \epsilon$. Thus

$$\left|\int \hat{f}_\alpha s\, d\bar{\mu}\right| = \left|\int f_\alpha s\, dm\right| < \epsilon.$$

But

$$\left|\int \hat{f}_\alpha s\, d\bar{\mu}\right| \geqslant \left|\int_F s\, d\bar{\mu}\right| - \left|\int_{O-F} \hat{f}_\alpha s\, d\bar{\mu}\right| \geqslant |\bar{\mu}(F)s| - \epsilon.$$

Therefore $|\bar{\mu}(F)s| \leqslant 2\epsilon$. Since $\epsilon > 0$ was arbitrary, we conclude that $\bar{\mu}(F)s = 0$ which proves the claim. Next we define μ: $Bo(X) \longrightarrow E'$ by $\mu(F \cap X) = \bar{\mu}(F)$ for each $F \in Bo(\beta X)$. If F_1, F_2 are Borel sets in βX such that $F_1 \cap X = F_2 \cap X$, then both $F_1 - F_2$ and $F_2 - F_1$ are disjoint from X and so $\bar{\mu}(F_1) = \bar{\mu}(F_1 \cap F_2) = \bar{\mu}(F_2)$. Hence μ is well defined. It is easy now to see that $\mu \in M_{\tau,p}(Bo(X), E')$. Moreover, it is clear that $\int f\, d\mu = \int \hat{f}\, d\bar{\mu} = \int \hat{f}\, d\bar{m} = \int f\, dm$ for all f in C_{rc}. Let $m_1 = \mu|_B$. Then $m_1 \in M(B, E')$ and $\int f\, dm = \int f\, dm_1$ for each f in C_{rc}. By Theorem 1.1, $m = m_1$ and hence μ is an extension of m. The theorem is proved.

The next theorem gives another characterization of the σ-additive and τ-additive members of $M(B, E')$. This characterization will be useful later. Let $m \in M(B, E')$. Define ϕ on $C(\beta X, E)$ by $\phi(\hat{f}) = \int f\, dm$, $f \in C_{rc}$. Then ϕ is continuous with respect to the uniform topology on $C(\beta X, E)$. Since $M(\beta X) = M_\tau(\beta X)$, there exists $\bar{m} \in M_{\tau,p}(Bo(\beta X), E')$ such that $\phi(\hat{f}) = \int \hat{f}\, d\bar{m}$ for each f in C_{rc}.

THEOREM 2.7. (a) $m \in M_\sigma(B, E')$ *iff* $\bar{m}_p(Z) = 0$ *for each zero set* Z *in* βX *which is disjoint from* X.

(b) m *is* τ*-additive iff* $\bar{m}_p(F) = 0$ *for each closed set* F *in* βX *which is disjoint from* X.

PROOF. (a) Assume that m is σ-additive. Let $s \in E$. For each $f \in C^b(X)$ we have $\int f\, d(ms) = \int fs\, dm = \int \hat{fs}\, d\bar{m} = \int \hat{f}\, d(\bar{m}s)$. Since ms is σ-additive, we have that $(\bar{m}s)(F) = 0$ for each Baire set F in βX which is disjoint from X (see Knowles [11, Theorem 2.1]). Let $\bar{\mu}$ be the restriction of $\bar{m}$ to $Ba(\beta X)$. Then $\mu \in M_{\sigma,p}(Ba(\beta X), E')$ and $\bar{\mu}_p = \bar{m}_p|_{Ba(\beta X)}$. By what we proved, $\bar{m}_p(F) = \bar{\mu}_p(F) = 0$ for each Baire set F in βX which is disjoint from X. Conversely, assume that $\bar{m}_p(Z) = 0$ for each zero set Z in βX disjoint from X. By regularity $\bar{\mu}_p(F) = 0$ for each Baire set F disjoint from X. Define μ: $Ba(X) \longrightarrow E'$ by $\mu(F \cap X) = \bar{\mu}(F)$ for each Baire set F in βX. This gives us a well-defined element of $M_{\sigma,p}(Ba(X), E')$. Moreover, if $m_1 = \mu|_{B(X)}$, then $\int f\, dm_1 = \int f\, d\mu = \int \hat{f}\, d\bar{\mu} = \int \hat{f}\, d\bar{m} = \int f\, dm$ for all f in C_{rc}. Thus $m_1 = m$ and hence m is σ-additive by 2.5.

(b) The proof is similar to that of (a).

THEOREM 2.8. *If we consider on* $C'_{rc} = M(B, E')$ *the weak topology* $\sigma(C'_{rc}, C_{rc})$, *then* $M_\sigma(B, E')$ *is sequentially closed.*

PROOF. Let $\{m_n\}$ be a sequence of elements of $M_\sigma(B, E')$, $m \in M(B, E')$, and assume that $m_n \longrightarrow m$. Let $s \in E$. For each f in $C^b(X)$ we have

$$\int f\, d(m_n s) = \int fs\, dm_n \longrightarrow \int fs\, dm = \int f\, d(ms).$$

Thus $m_n s \longrightarrow ms$ in the $\sigma(M(X), C^b)$ topology. By Aleksandrov [1], ms is σ-additive. This, being true for all $s \in E$, implies that m is σ-additive.

3. **A weighted type topology on C_{rc}.** Let V be a family of bounded continuous real-valued functions on X. Assume that V has the following two properties:

(1) For each x in X there exists $h \in V$ with $h(x) \neq 0$.

(2) Given u, v in V and a positive number d, there exists w in V with $|w| \geqslant du, dv$ (pointwise). We will denote by w_V the locally convex topology on C_{rc} generated by the family of seminorms $\{\|\cdot\|_{p,h}: p \in I, h \in V\}$ where $\|\cdot\|_{p,h}$ is defined on C_{rc} by

$$\|f\|_{p,h} = \sup\{p(h(x)f(x)): x \in X\} = \|hf\|_p.$$

It is clear that w_V has a base at zero consisting of all sets of the form $\{f \in C_{rc}: \|hf\|_p \leqslant 1\}$ where $h \in V$ and $p \in I$. It is also clear that w_V is Hausdorff and that $w_V \leqslant \sigma$. Hence $(C_{rc}, w_V)' \subset (C_{rc,\sigma})' = M(B, E')$. We will identify the dual space of (C_{rc}, w_V). We begin with an easily established lemma.

LEMMA 3.1. *Let $m \in M(B, E')$ and $h \in C^b$. For each F in B we define $\mu(F)$ on E by $\mu(F)s = \int_F hd(ms)$. Then $\mu \in M(B, E')$ and $\int f\, d\mu = \int fh\, dm$ for each f in C_{rc}.*

We denote the element $\mu \in M(B, E')$, defined in 3.1, by hm. Let

$$V \cdot M(B, E') = \{hm: h \in V, m \in M(B, E')\}.$$

We will prove the following.

THEOREM 3.2. *The space $(C_{rc}, w_V)'$ is isomorphic to the space $V \cdot M(B, E')$ and the isomorphism $\phi \longrightarrow m$ is given by the formula, where $\phi(f) = \int f\, dm$ for all $f \in C_{rc}$.*

Set $H = (C_{rc}, w_V)'$.

LEMMA 3.3. *If $h \in V$ and $m \in M_p(B, E')$, then hm gives an element of the dual space of (C_{rc}, w_V).*

PROOF. Set $\mu = hm$. The set $W = \{f: \|fh\|_p \leqslant 1\}$ is a w_V-neighborhood of zero. Moreover, if $f \in W$, then

$$\left|\int f\, d\mu\right| = \left|\int fh\, dm\right| \leqslant \|fh\|_p \|m\|_p \leqslant \|m\|_p.$$

This completes the proof.

LEMMA 3.4. *Let $h \in V$ and define $T_h = T: C_{rc} \longrightarrow C_{rc}$, $Tf = hf$. Then T is $\sigma(C_{rc}, H) - \sigma(C_{rc}, C'_{rc})$ continuous.*

Moreover, if T' is the adjoint of T and if $p \in I$, then $T'(B_p^\circ) = W_p^\circ$, where $B_p = \{f \in C_{rc}: \|f\|_p \leqslant 1\}$, $W_p = T^{-1}(B_p)$, B_p° the polar of B_p with respect to the pair $\langle C_{rc}, C'_{rc}\rangle$, and W_p° the polar of W_p with respect to the pair $\langle C_{rc}, H\rangle$.

PROOF. Let $\{f_\alpha\}$ be a net in C_{rc} converging to zero in the $\sigma(C_{rc}, H)$ topology. Let $m \in M(B, E')$. In view of 3.3 we have $\int f_\alpha d(hm) \longrightarrow 0$. Thus $\int hf_\alpha dm \longrightarrow 0$ which shows that $Tf_\alpha \longrightarrow 0$ in the $\sigma(C_{rc}, C'_{rc})$ topology. Thus T is $\sigma(C_{rc}, H) - \sigma(C_{rc}, C'_{rc})$ continuous. Therefore T' exists and $T'(C'_{rc}) \subset H$. Also T' is $\sigma(C'_{rc}, C_{rc}) - \sigma(H, C_{rc})$ continuous. The set B_p is clearly σ-closed. Since B_p is convex and since σ and $\sigma(C_{rc}, C'_{rc})$ are both compatible with the pair $\langle C_{rc}, C'_{rc}\rangle$, B_p is $\sigma(C_{rc}, C'_{rc})$ closed. Also B_p is balanced. Thus $B_p = B_p^{\circ\circ}$ by the bipolar theorem (see Schaefer [14, p. 126]). Let $W = [T'(B_p^\circ)]^\circ$. If $f \in W$ and $m \in B_p^\circ$, then $|\langle m, Tf\rangle| = |\langle T'm, f\rangle| \leqslant 1$. This shows that $Tf \in B_p^{\circ\circ} = B_p$. Hence $W \subset W_p$. On the other hand, if $f \in W_p$ and $m \in B_p^\circ$, then

$|\langle f, T'm\rangle| = |\langle Tf, m\rangle| \leq 1$. Thus $W_p \subset W$ and so $W = W_p$. The set B_p° is $\sigma(C'_{rc}, C_{rc})$ compact by the Alaoglu theorem (see Köthe [10, p. 248]). Hence $T'(B_p^\circ)$ is $\sigma(H, C_{rc})$ compact. Also $T'(B_p^\circ)$ is convex and balanced. Therefore, by the bipolar theorem, $T'(B_p^\circ) = [T'(B_p^\circ)]^{\circ\circ} = W_p^\circ$. The lemma is proved.

LEMMA 3.5. *If $\phi \in (C_{rc}, w_V)'$, then there exists $h \in V$, $m \in M(B, E')$ such that $\phi(f) = \int f\, d(hm)$ for all $f \in C_{rc}$.*

PROOF. Since ϕ is w_V-continuous, there exist $h \in V$ and $p \in I$ such that $W_p = \{f\colon \|hf\|_p \leq 1\} \subset \{f\colon |\phi(f)| \leq 1\}$. Let $T = T_h$ be as in Lemma 3.4. In view of 3.4, we have $T'(B_p^\circ) = W_p^\circ$. Since $\phi \in W_p^\circ$ there exists $m \in B_p^\circ$ such that $\phi = T'm$. Now, for each $f \in C_{rc}$, we have $\langle f, \phi\rangle = \langle f, T'm\rangle = \langle Tf, m\rangle = \int f\, d(hm)$. This completes the proof.

Combining 3.3 and 3.5 we get Theorem 3.2.

4. **The strict and superstrict topologies on C_{rc}.** Buck defined in [4] the strict topology on the space of bounded continuous functions on a locally compact space and he identified the dual in the scalar case. The dual space for the vector case was studied by Wells [18]. Recently Sentilles [15] and Fremlin-Garling-Haydon [5] defined the strict and superstrict topologies on the space of all bounded continuous real-valued functions on a completely regular Hausdorff space. They identified the strict and superstrict dual of C^b with the spaces $M_\tau(X)$ and $M_\sigma(X)$ respectively. These and other authors completed the result of Hewitt [6] on the representation of linear functionals on spaces of continuous functions. In [3] Bogdanowicz studied the space of continuous linear functionals on the space of continuous mappings from a compact space into a locally convex space. In this section we will introduce on C_{rc} two locally convex topologies β_1 and β which yield as dual spaces the spaces of all σ-additive and all τ-additive members of $M(B, E')$ respectively. Our approach will be analogous to that of Sentilles.

Let Ω (Ω_1) denote the collection of all closed (zero) sets in βX which are disjoint from X. For Q in Ω, let $B_Q = \{h \in C^b\colon \hat{h} = 0 \text{ on } Q\}$. Clearly B_Q has all the properties of the family V mentioned in the beginning of §3.

Let β_Q be the locally convex topology on C_{rc} generated by the family of seminorms $f \longrightarrow \|hf\|_p$, $h \in B_Q$, $p \in I$. The strict topology β on C_{rc} is defined to be the inductive limit of the topologies β_Q, $Q \in \Omega$. The superstrict topology β_1 on C_{rc} is the inductive limit of the topologies β_Z, $Z \in \Omega_1$. If π is the pointwise convergence topology, one can easily verify the following

THEOREM 4.1. $\pi \leq \beta \leq \beta_1 \leq \sigma$.

THEOREM 4.2. *$\beta = \sigma$ iff X is compact.*

PROOF. Clearly $\beta = \sigma$ if X is compact. On the other hand assume that X is not compact and that $\beta = \sigma$.

Let $x \in \beta X - X$, $Q = \{x\}$. Let $p \in I$, $s \in E$ be such that $p(s) = 2$. Set $W = \{fC_{rc}: \|f\|_p \leqslant 1\}$. Then W is a σ-neighborhood of zero. By hypothesis W is also a β-neighborhood of zero. Since $\beta \leqslant \beta_Q$, W is a β_Q-neighborhood of zero. Thus there exist $h \in B_Q$ and p_1 in I such that $V = \{fC_{rc}: \|hf\|_{p_1} \leqslant 1\} \subset W$. Choose $\delta > 0$ such that $\delta p_1(s) \leqslant 1$, and set $F = \{y \in \beta X: |\hat{h}(y)| \geqslant \delta\}$.

Let $g \in C^b$, $0 \leqslant g \leqslant 1$, $\hat{g}(x) = 1$ and $\hat{g} = 0$ on F. But then the function $f = gs$ is in V but not in W. This contradiction completes the proof.

Since X is pseudocompact iff $\Omega_1 = \{\varnothing\}$, we have the following theorem for β_1 whose proof is similar to that of Theorem 4.2.

THEOREM 4.3. *$\beta_1 = \sigma$ iff X is pseudocompact.*

If X is locally compact, then X is open in βX. Let $Q = \beta X - X$. Then B_Q is the space of all continuous real functions on X that vanish at infinity. Hence, as one can easily prove, $\beta = \beta_Q$ coincides with the strict topology as defined by Buck in [4]. We will next identify the dual spaces of (C_{rc}, β) and (C_{rc}, β_1).

LEMMA 4.4. *If $\phi \in (C_{rc}, \beta)'$, then there exists $m \in M_\tau(B, E')$ such that $\phi(f) = \int f\, dm$ for all $f \in C_{rc}$.*

PROOF. Since $\beta \leqslant \sigma$ there exists $m \in M(B, E')$ such that $\phi(f) = \int f\, dm$ for all f in C_{rc}. Let $\bar{m} \in M_{\tau,p}(Bo(\beta X), E')$ be such that $\phi(f) = \int \hat{f} d\bar{m}$ for all f in C_{rc}. Let $Q \in \Omega$. Since ϕ is β_Q-continuous, there exists (by 3.5) $h \in B_Q$ and $\mu \in M(B, E')$ such that $\int hf\, d\mu = \phi(f)$ for all $f \in C_{rc}$. Let $\bar{\mu} \in M_\tau(Bo(\beta X), E')$ be such that $\int f\, d\mu = \int \hat{f}\, d\bar{\mu}$ for all $f \in C_{rc}$. Then $\int \hat{f}\, d\bar{m} = \phi(f) = \int hf\, d\mu = \int \hat{h}\hat{f}\, d\bar{\mu}$ for each $f \in C_{rc}$. It follows that $\bar{m} = \hat{h}\bar{\mu}$. If F is a Borel set in βX contained in Q and if $s \in E$, then $\bar{m}(F)s = \int_F \hat{h}\, d(\bar{\mu}s) = 0$. We conclude that $\bar{m}_p(Q) = 0$. This, being true for all $Q \in \Omega$, implies that m is τ-additive by 2.7. This completes the proof.

LEMMA 4.5. *If $m \in M_{\tau,p}(B, E')$, then the map $\phi_m: C_{rc} \longrightarrow R$, $\phi_m(f) = \int f\, dm$, is β-continuous.*

PROOF. It suffices to show that ϕ_m is β_Q-continuous for every Q in Ω. So, let $Q \in \Omega$. Define T: $C^b \longrightarrow R$ by $T(f) = \int f\, dm_p$. Since m_p is τ-additive, T is $\beta(C^b)$ continuous, where $\beta(C^b)$ is the strict topology on C^b as defined by Sentilles in [15] (see Sentilles, Theorem 4.3). Hence there exists g in B_Q such that

$$W = \{f \in C^b: \|gf\| \leqslant 1\} \subset \{f \in C^b: |T(f)| \leqslant 1\}.$$

Set $V = \{f \in C_{rc}: \|gf\|_p \leqslant 1\}$ and let $f \in V$.

Define $h: X \longrightarrow R$, $h(x) = p(f(x))$. Clearly $h \in C^b$. Moreover $|h(x)g(x)| = p(g(x)f(x)) \leqslant 1$ for all $x \in X$ and hence $h \in W$. It follows that $|\int f\,dm| \leqslant \int h\,dm_p = T(h) \leqslant 1$. This shows that ϕ_m is β continuous. The lemma is proved.

Combining Lemmas 4.4 and 4.5 we get

THEOREM 4.6. *The space $M_\tau(B, E')$ is isomorphic to the space $(C_{rc}, \beta)'$ via the isomorphism $m \longmapsto \phi_m$ where $\phi_m(f) = \int f\,dm$ for all f in C_{rc}.*

Using similar arguments we prove

THEOREM 4.7. *The space $M_\sigma(B, E')$ is isomorphic to the space (C_{rc}, β_1) via the isomorphism $m \longmapsto \phi_m$, $\phi_m(f) = \int f\,dm$.*

5. **The case of a locally convex lattice E.** In this section E will be assumed to be a locally convex lattice. By Peressini [13, p. 105] there exists a generating family of continuous seminorms p such that $|x| \leqslant |y|$ implies $p(x) \leqslant p(y)$. In view of this, we may assume that every $p \in I$ has the above property. The space (C_{rc}, σ) is, under the pointwise ordering, a locally convex lattice. The question we are going to investigate now is the following: Which elements of C'_{rc} correspond to members of $M_\sigma(B, E')$ and which to members of $M_\tau(B, E')$? We will show that these are exactly the σ-additive and τ-additive members of C'_{rc}.

DEFINITION. For a net $\{f_\alpha\}$ in C_{rc} we say that $\{f_\alpha\}$ decreases to zero, and write $f_\alpha \downarrow 0$, if for each $x \in X$ we have $\lim f_\alpha(x) = 0$ and $0 \leqslant f_\alpha(x) \leqslant f_\gamma(x)$ if $\alpha \geqslant \gamma$. We define similarly what we mean by saying that a sequence $\{f_n\}$ in C_{rc} decreases to zero. An element ϕ of C'_{rc} is called σ-additive if $\lim \phi(f_n) = 0$ for each sequence $\{f_n\}$ in C_{rc} that decreases to zero. An element ϕ of C'_{rc} is called τ-additive if $\lim \phi(f_\alpha) = 0$ whenever $f_\alpha \downarrow 0$. We will denote by $L_\sigma(C_{rc})$ and $L_\tau(C_{rc})$ the spaces of all σ-additive and all τ-additive members of C'_{rc} respectively.

THEOREM 5.1. *Let $\phi \in C_{rc}$. Then ϕ is τ-additive iff there exists $m \in M_\tau(B, E')$ such that $\phi(f) = \int f\,dm$ for all $f \in C_{rc}$.*

PROOF. Let $m \in M_{\tau,p}(B, E')$ be such that $\phi(f) = \int f\,dm$ for all $f \in C_{rc}$. Let $\{f_\alpha\}$ be a net in C_{rc} that decreases to zero. For each α, let $h_\alpha: X \longrightarrow R$, $h_\alpha(x) = p(f(x))$. Since p has the property that $p(s) \leqslant p(t)$ whenever $|s| \leqslant |t|$, it follows that $h_\alpha \downarrow 0$. Hence $|\int f_\alpha\,dm| \leqslant \int h_\alpha\,dm_p \longrightarrow 0$ since m_p is τ-additive (see Varadarajan [17, p. 174]).

Conversely, assume that ϕ is τ-additive. Let $s \geqslant 0$, $s \in E$. If $\{f_\alpha\}$ is a net in $C^b(X)$ which decreases to zero, then $f_\alpha s \downarrow 0$. Hence $\int f_\alpha\,d(ms) = \int f_\alpha s\,dm = \phi(f_\alpha s) \longrightarrow 0$. It follows that ms is τ-additive (see Varadarajan [17, p. 174]).

Since every element of E is the difference of two positive elements, it follows that $ms \in M_\tau(X)$ for all $s \in E$ and hence m is τ-additive.

Using an analogous argument we prove the following:

THEOREM 5.2. *Let $\phi \in C'_{rc}$. Then ϕ is σ-additive iff there exists $m \in M_\sigma(B, E')$ such that $\phi(f) = \int f\,dm$ for all f in C_{rc}.*

LEMMA 5.3. *Let $m \in M_p(B, E')$ and $|m| = \sup(m, -m)$. Then $|m| \in M_p(B, E')$ and $|m|_p = m_p$.*

PROOF. Recall that p has the property that $p(s) \leqslant p(t)$ whenever $|s| \leqslant |t|$. As shown in [8], for each $s \geqslant 0$ in E and each F in $B(X)$ we have $|m|(F)s = \sup \Sigma |m(F_i)|s$ where the supremum is taken over all finite B-partitions $\{F_i\}$ of F. Let now $F \in B(X)$. If $F_1, \cdots, F_n$ is a B-partition of F, and if $s_i \in E$ with $p(s_i) \leqslant 1$,

$$\left|\sum m(F_i)s_i\right| \leqslant \sum |m(F_i)|\,|s_i| \leqslant \sum |m|(F_i)|s_i| \leqslant |m|_p(F)$$

since $p(|s_i|) = p(s_i) \leqslant 1$. Thus $m_p(F) \leqslant |m|_p(F)$. On the other hand, let $G_1, \cdots, G_n$ be a B-partition of F and let $s_i \in E$ with $p(s_i) \leqslant 1$. We will show that $|\Sigma |m|(G_i)s_i| \leqslant m_p(F)$. Since $p(|s_i|) \leqslant 1$ and since $|\Sigma |m|(G_i)s_i| \leqslant \Sigma |m|(G_i)|s_i|$, we may assume that $s_i \geqslant 0$. Let $\epsilon > 0$ be given. For each i, $1 \leqslant i \leqslant n$, there exists a B-partition $F_1^i, \cdots, F_{K_i}^i$ of G_i such that

$$\sum_{j=1}^{K_i} |m(F_j^i)|s_i > |m|(G_i)s_i - \frac{\epsilon}{n}.$$

Let $N = K_1 + \cdots + K_n$. Choose $t_{ij} \in E$, $|t_{ij}| \leqslant s_i$, such that $|m(F_j^i)t_{ij}| > |m(F_j^i)|s_i - \epsilon/N$. Since $p(t_{ij}) \leqslant 1$ and $\{F_j^i\}$ is a B-partition of F, we have

$$m_p(F) \geqslant \sum_{i,j} |m(F_j^i)t_{ij}| = \sum_{i=1}^{n} \sum_{j=1}^{K_i} |m(F_j^i)t_{ij}|$$

$$\geqslant \sum_{i=1}^{n} \sum_{j=1}^{K_i} |m(F_j^i)|s_i - \epsilon \geqslant \sum_{i=1}^{n} |m|(G_i)s_i - 2\epsilon.$$

Since $\epsilon > 0$ was arbitrary, we have $\Sigma |m|(G_i)s_i \leqslant m_p(F)$. This proves that $|m|_p(F) \leqslant m_p(F)$ and the lemma is proved.

By the above lemma, if $m \in M_\tau(B, E')$, then $|m|$ is also τ-additive. From this follows that $M_\tau(B, E')$ is an ideal in $M(B, E')$. Since the map $m \longrightarrow \phi_m$, of Theorem 1.1, is lattice-preserving and since $M_\tau(B, E')$ corresponds to $L_\tau(C_{rc})$ in this map, it follows that $L_\tau(C_{rc})$ is an ideal in the Riesz space C'_{rc}. The same is true for the space $L_\sigma(C_{rc})$. We have thus the following theorem.

THEOREM 5.4. *Each of the spaces $L_\sigma(C_{rc})$ and $L_\tau(C_{rc})$ is an ideal in the Riesz space C'_{rc}.*

It is well known (Knowles [11, p. 149]) that any positive linear functional ϕ on C^b can be written uniquely as a sum of a positive purely finitely-additive functional on C^b, a positive purely σ-additive and a positive τ-additive functional on C^b. It is therefore natural to ask whether this is true in our space (C_{rc}, σ). We will show that the answer to this question is affirmative.

DEFINITION. An element $\phi \geqslant 0$ in C'_{rc} is called purely finitely-additive if the only σ-additive functional ϕ_1 in C_{rc} with $0 \leqslant \phi_1 \leqslant \phi$ is the zero functional. Similarly, an element $\phi \geqslant 0$ of $L_\sigma(C_{rc})$ is purely σ-additive if $0 \leqslant \phi_1 \leqslant \phi$ and $\phi_1 \in L_\tau(C_{rc})$ implies that $\phi_1 = 0$.

We are going to prove the following

THEOREM 5.5. *Given $\phi \geqslant 0$ in C'_{rc} there are ϕ_1, ϕ_2, ϕ_3 in C'_{rc}, ϕ_1 purely finitely-additive, ϕ_2 purely σ-additive, ϕ_3 τ-additive, $\phi_1, \phi_2, \phi_3 \geqslant 0$, such that $\phi = \phi_1 + \phi_2 + \phi_3$. Moreover this decomposition is unique.*

To begin with, assume that $\phi, \phi_1, \phi_2 \geqslant 0$ in C'_{rc}, ϕ_1 σ-additive, $\phi = \phi_1 + \phi_2$. Let $m, m_1, m_2 \in M_{\tau,p}(Bo(\beta X), E')$ be such that $\phi(f) = \int \hat{f}\, dm$, $\phi_i(f) = \int \hat{f}\, dm_i$, $i = 1, 2$, for all $f \in C_{rc}$.

Set $d = \inf\{m_p(V)\colon V \supset X,\ V$ a cozero set in $\beta X\}$. Choose a decreasing sequence $\{U_i\}$ of cozero sets in βX, $X \subset U_i$, such that $m_p(U_i) \longrightarrow d$. Let $K = \bigcap U_i$. Clearly $m_p(K) = d$. Since ϕ_1 is σ-additive we have $(m_1)_p(\beta X - K) = 0$ by 2.7. Hence $m_1(F) = m_1(F \cap K) \leqslant m(F \cap K)$ for each Borel set F in βX since $m_2 \geqslant 0$ and $m = m_1 + m_2$. Define m_3 on $Bo(\beta X)$ by $m_3(F) = m(F \cap K)$. Then $m_3 \in M_{\tau,p}(Bo(\beta X), E')$ and $(m_3)_p(F) = m_p(F \cap K)$ for each F in $Bo(\beta X)$. Since E is locally solid, the positive cone is closed (see Schaefer [14, p. 235]). Now let $f \geqslant 0$ in C_{rc}. If $y \in \beta X$ and if $\{x_\alpha\}$ is a net in X converging to y in βX, then $\hat{f}(y) = \lim \hat{f}(x_\alpha) = \lim f(x_\alpha) \geqslant 0$. Thus $\hat{f} \geqslant 0$. It follows that the map $\phi_3\colon C_{rc} \longrightarrow R$, $\phi_3(f) = \int \hat{f}\, dm_3$ is positive. Moreover $\phi_3 \geqslant \phi_1$ since $m_3 \geqslant m_1$. We next show that ϕ_3 is σ-additive. Indeed, let Z be a zero set in βX disjoint from X. Then $U = \beta X - Z$ is a cozero set containing X. Since $X \subset U \cap U_i \downarrow K \cap U$, we have

$$d \leqslant \lim m_p(U \cap U_i) = m_p(K \cap U) \leqslant m_p(K) = d.$$

So $(m_3)_p(\beta X) = m_p(K) = m_p(K \cap U) = (m_3)_p(U)$ and hence $(m_3)_p(Z) = 0$. By 5.2 and 2.7 ϕ_3 is σ-additive. Clearly $\phi - \phi_3$ is purely finitely-additive. We have thus proved

THEOREM 5.6. *If $\phi \geqslant 0$ in C'_{rc}, then there are unique $\phi_1, \phi_2 \geqslant 0$ in C'_{rc}, ϕ_1 purely finitely-additive and ϕ_2 σ-additive, such that $\phi = \phi_1 + \phi_2$.*

Next assume that $0 \leqslant \phi \in L_\sigma(C_{rc})$. Suppose that $\phi = \phi_1 + \phi_2$, $\phi_1 \in L_\tau(C_{rc})$, $\phi_2 \in L_\sigma(C_{rc})$, $\phi_1, \phi_2 \geqslant 0$. Let $m, m_1, m_2 \in M_{\tau,p}(Bo(\beta X), E')$ be such that $\phi(f) = \int \hat{f}\, dm$, $\phi_i(f) = \int \hat{f}\, dm_i$, $i = 1, 2$, for all f in C_{rc}. Let $d = \inf\{m_p(O): O$ open in βX, $X \subset O\}$. Choose a decreasing sequence $\{O_n\}$ of open sets in βX, $X \subset O_n$, $m_p(O_n) \longrightarrow d$. Since ϕ_1 is τ-additive we have that $(m_1)_p(\beta X - K) = 0$. Thus $m_1(F) = m_1(F \cap K) \leqslant m(F \cap K)$ for all Borel sets F in βX. Define m_3 on $Bo(\beta X)$ by $m_3(F) = m(F \cap K)$. Then $m_3 \in M_{\tau,p}(Bo(\beta X), E')$ and $m_3 \geqslant m_1$. Let Q be a closed set in βX disjoint from X. Then $O = \beta X - Q \supset X$ and O is open. Hence $d \leqslant \lim m_p(O \cap O_i) = m_p(O \cap K) \leqslant m_p(K) = d$. It follows that $(m_3)_p(\beta X) = (m_3)_p(O)$ and hence $(m_3)_p(O) = 0$. This, being true for all closed sets in βX which are disjoint from X, implies that the map $\phi_3: C_{rc} \longrightarrow R$, $\phi_3(f) = \int \hat{f}\, dm_3$, is τ-additive. Also $\phi_3 \geqslant 0$. Moreover, $\phi - \phi_3$ is purely σ-additive. We have thus proved

THEOREM 5.7. *Given $\phi \geqslant 0$ in $L_\sigma(C_{rc})$, there are unique $\phi_1, \phi_2 \geqslant 0$, $\phi_1 \in L_\tau(C_{rc})$, ϕ_2 purely σ-additive, such that $\phi = \phi_1 + \phi_2$.*

Combining Theorems 5.6 and 5.7 we get Theorem 5.5.

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